

Gauss's law

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“But still try, for who knows what is possible?”

— Michael Faraday (1791-1867)

Coulomb's law for a point charge q is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} q \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad (1)$$

Note that we can define $|\mathbf{x} - \mathbf{x}'| = r$ for simplicity of notation. If we put the charge at the origin, we obtain

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}. \quad (2)$$

In the above equation we dot both sides with the unit normal $\hat{\mathbf{n}}$ and obtain

$$\begin{aligned} \mathbf{E} \cdot \hat{\mathbf{n}} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \cos \theta \end{aligned} \quad (3)$$

Now perform the surface integral over the closed surface

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}} da = \oint \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \cos \theta da \quad (4)$$

Now using the relation $\cos \theta da = r^2 d\Omega$ (using solid angle definition), we write the above

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equation as

$$\begin{aligned}
 \oint \mathbf{E} \cdot \hat{\mathbf{n}} da &= \oint \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} r^2 d\Omega \\
 &= \oint \frac{1}{4\pi\epsilon_0} q d\Omega \\
 &= \frac{1}{4\pi\epsilon_0} q \oint d\Omega \\
 &= \frac{1}{4\pi\epsilon_0} q 4\pi \\
 &= \frac{q}{\epsilon_0}
 \end{aligned} \tag{5}$$

or

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}}. \tag{6}$$

This is the Gauss's law for a point charge placed at the origin. If we consider many charges scattered about instead of a single charge, we can write using the superposition principle or

$$\mathbf{E} = \sum_{i=1}^n \mathbf{E}_i. \tag{7}$$

Now Eq. (6) for many charges $\{q_i\}$ for $i = 1, \dots, n$, enclosed within the surface can be written as

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^n \oint \mathbf{E}_i \cdot d\mathbf{a} = \sum_{i=1}^n \frac{q_i}{\epsilon_0} \tag{8}$$

or

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{inc}}{\epsilon_0}}, \tag{9}$$

where Q_{inc} denotes the total charge enclosed within the surface. For continuous charge distribution, \mathbf{E} will be given by

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_0} \int dq \\
 &= \frac{1}{\epsilon_0} \int \frac{dq}{d\mathbf{x}} d\mathbf{x} \quad (d\mathbf{x} \equiv dx dy dz)
 \end{aligned}$$

or

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho(\mathbf{x}) d\mathbf{x} \tag{10}$$

Next, using divergence theorem [*]

$$\oint_s \mathbf{V} \cdot d\mathbf{a} = \int_v \nabla \cdot \mathbf{V} d\tau, \quad (d\tau \equiv dx dy dz) \quad (11)$$

we can write Eq. (10) as

$$\int \nabla \cdot \mathbf{E} d\mathbf{x} = \frac{1}{\epsilon_0} \int \rho(\mathbf{x}) d\mathbf{x}$$

or

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\epsilon_0}}. \quad (12)$$

[1] D.J. Griffiths, *Introduction to Electrodynamics*, New Jersey: Prentice Hall, (1999) .

[2] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, (1962).

[*] For details of divergence theorem, see B.Sc. Part-III Mathematical Physics notes uploaded on the website